On the relaxation of fluctuations in the steady state of the Stratonovich model

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Abstract. A detailed investigation has been carried out of the relaxation of fluctuations in the steady state of the Stratonovich model, also known as the random growing rate model (RGRM). The autocorrelation time of this system, driven by parametric white noise in the linear term, has been measured for an electronic circuit model, computed by digital simulation and calculated by use of a continued fraction expansion method. The results are all consistent with each other, and with those obtained from the matrix continued fraction method of Jung and Risken, provided that explicit account is taken of the ways in which any macroscopic real physical system is bound to differ from the idealisation represented by the original form of the Stratonovich model. In particular, it is necessary to recognise the existence of weak additive noise and a small additive constant. The physical origins of some earlier, seemingly discrepant, calculations and experimental data are discussed.

1. Introduction

A useful way in which to characterise the relaxation of fluctuations from the stationary state of a stochastic non-linear system is by specification of its correlation time, T. The Stratonovich model (also known as the random growing rate model, or RGRM) has been one of the most intensively studied of all such systems. Yet, until very recently, there remained a considerable measure of uncertainty as to the form and magnitude to be expected of T under strong external forcing by white noise; it has been even less clear whether or not the sorts of real physical systems expected to be described by the model would actually behave in the manner predicted by the theory.

The stochastic differential equation describing the model in question, which was first studied by Stratonovich (1967) and subsequently by many other workers (e.g., Schenzle and Brand 1979, Brenig and Banai 1982, Graham and Schenzle 1982, Sancho et al 1982a, b, Hernández-Machado et al 1984, Faetti et al 1984, Horsthemke and Lefever 1984, Risken 1984, Jung and Risken 1985, Mannella et al 1986), may be written

$$\dot{x} = dx - bx^3 + \xi x \tag{1}$$

where ξ represents Gaussian white noise with autocorrelation function

$$\langle \xi(t)\xi(t')\rangle = 2Q\delta(t - t') \tag{2}$$

and d and b are constants. In what follows we will assume, except where explicitly stated to the contrary, that $d = b = 1$. All the measurements and calculations to be reported will be normalised by appropriate changes of variable so that this is the case,

thereby permitting more convenient comparison of the different results. We will be mainly concerned with the correlation time, T, which is defined as

$$T = \int_0^x c(s)\,\mathrm{d}s/c(0) \tag{3}$$

where

$$c(s) = \langle \delta x(t+s)\delta x(t) \rangle \tag{4}$$

is the autocorrelation function, and

$$\delta x(t) = x(t) - \langle x(t) \rangle \tag{5}$$

represents the departure of x from its mean value.

Following the calculations recently reported by Jung and Risken (1985, hereafter referred to as JR), the variation of T with Q for (1) may be regarded as a completely solved problem. By application of their matrix continued fraction (MCF) technique (and also by directly integrating (3), working on the Fokker–Planck equation corresponding to (1)) they were able to show that, as Q increases from zero, T^{-1} falls monotonically from its deterministic $(Q \to 0)$ value of 2 towards an asymptotic $(Q \to \infty)$ limit of $2/\pi$. Earlier calculations (Graham and Schenzle 1982) based on a method of linear embedding were in good agreement with this result in the limits of large and small Q; a digital simulation (Sancho *et al* 1982a) yielded values of $T^{-1}(Q)$ that were systematically high, but nonetheless in good qualitative agreement over a wide range of Q. Calculations (Hernández-Machado *et al* 1984, Faetti *et al* 1984) based on the method of continued fraction expansion (CFE) can now be seen to have yielded correct answers for small Q, but erroneous results for larger Q, where it was found that T^{-1} passed through a minimum and then increased with further increase of Q. An analogue simulation (Faetti *et al* 1984) of (1) appeared to confirm the results of the CFE calculations at large Q, where a monotonic increase of T^{-1} with Q was observed, in clear disagreement with JR.

In the present paper we treat two distinct, but closely related, questions arising from the disagreements and inconsistent results obtained in the earlier work. First, can one expect T^{-1} for a real macroscopic physical system to behave in the manner predicted by JR on the basis of (1)? We address this question experimentally in § 2 by describing studies of the behaviour of a new electronic circuit model of (1), a circuit that is different in design philosophy and mode of operation from one (Faetti *et al* 1984) reported previously. Such circuits, though not of course naturally occurring systems, are nonetheless real physical systems in their own right. In common with nature itself, they possess 'non-idealities' as compared with theoretical models; their study can help towards an understanding of which non-idealities are important and what their effects are on the behaviour of the system. They can thus provide illuminating demonstrations of the applicability, or otherwise, of equations such as (1) to the real world. The second question that we wish to consider is this: why did the CFE calculations, apparently correctly executed, lead to what can now be seen as erroneous results, and is it possible to make the method more reliable? We will show in § 3 that the earlier disagreements were due to an improper use of the CFE technique and that, with a modified application of the method, much better agreement with JR may be obtained.

In § 4 we describe a new digital simulation of (1), leading to results that are more precise and in better agreement with JR than those (Sancho *et al* 1982a) reported previously.

The values of $T^{-1}(Q)$ obtained by these three different methods are compared with each other and with JR in § 5, and the probable reasons for some of the observed discrepancies are discussed. The principal conclusions are summarised in § 6. A letter (Mannella *et al* 1986) presenting some preliminary results of the research has already been published.

2. The analogue experiment

2.1. *Electronic circuit and data analysis*

The electronic circuit used for the analogue measurements was similar in its general design philosophy and mode of operation to those described previously (Smythe *et al* 1983, Sancho *et al* 1985, Mannella *et al* 1986). It was based on two analogue multipliers and two operational amplifiers, arranged so as to execute the arithmetic operations shown in figure 1.
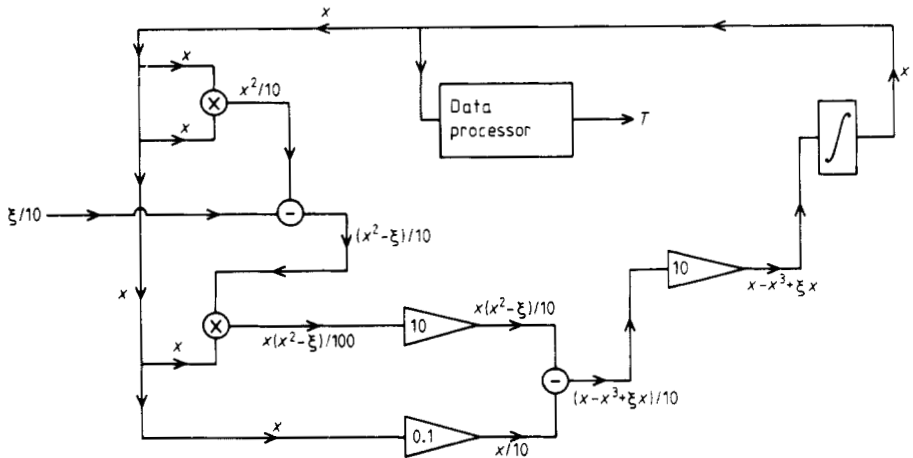

Figure 1. Block diagram of the electronic circuit designed to model equation (1). The data processor computes the relaxation time T of the noisy voltage $x(t)$, but does not affect the operation of the circuit in any way.

The circuit incorporated some improvements, as compared to the prototype version used for our preliminary measurements. In particular, it was able to accommodate a larger dynamic range of ξ and it suffered to a smaller extent from the parameter drift that was found to be of unexpectedly great significance in the initial experiments. Not shown in the block diagram are the trimmers that were used to check that the multipliers and operational amplifiers gave an accurately zero ($< \pm 1$ mV) output for zero input. The three terms in (1) are built up in stages, as shown, with some of the arithmetic operations being carried out on scaled ($\times 0.1$ or 0.01) values of x so as to minimise the danger of exceeding the dynamic range of any of the components; once the separate terms have been combined, however, the overall scale factor is returned to unity once more, so that $x = 1$ implies 1 V in the circuit. The sum of terms is integrated with a time constant $\tau_1 = 2 \times 10^4$ μs to produce x, which is then returned to the input, representing the equality in (1).

The noise input was taken from a Wandel and Goltermann RG1 Gaussian noise generator and filtered such that it was exponentially correlated with a correlation time $\tau_N = 50 \ \mu s$; the circuit was arranged such that the noise was added at a scale factor of 0.1, thus reducing problems of possible 'clipping' at large values of Q. Because $\tau_1 \gg \tau_N$, the noise is perceived by the circuit as white; the effective value of Q for any given root mean square noise voltage, V_{rms}, is given by

$$Q = V_{rms}^2 \frac{\tau_N}{\tau_1}. \tag{6}$$

The fluctuating $x(t)$ is taken to a Nicolet 1280 data processor for analysis; it must be emphasised that the data processor was used purely as an analytical instrument and that it in no way affected the operation of the circuit. Blocks of 1024 discrete samples of $x(t)$ were digitised and their correlation functions $c(s)$ determined (Sancho et al 1985) by a standard fast Fourier transform technique (Beauchamp and Yuen 1979). The correlation functions of successive blocks of $x(t)$ were summation averaged until the statistical quality of the result was considered adequate, typically incorporating 1000 blocks on average. The time taken for this procedure was an order of magnitude less than in our preliminary measurements, based on a Nicolet 1080 data processor, because of the use of a fast internal FFT co-processor within the Nicolet 1280. A small-baseline correction was applied to the computed $c(s)$ so that the flat region of the correlation function at large s coincided accurately with zero, and the area under the curve was then measured to find T.

2.2. Experimental results

Preliminary measurements for the new circuit showed that, just occasionally, the average value of x would jump from a positive to a negative value or vice versa (Faetti et al 1982, Hanggi et al 1985). Such behaviour cannot, of course, occur for the original Stratonovich model of (1); it is a consequence of the very weak additive noise present in the electronic circuit. Although the events in question were extremely infrequent, they exerted a disproportionately large influence on the final averaged value of T, increasing it to a significant extent. Consequently, the data analysis algorithm was designed so as to ignore any $x(t)$ blocks where a change of sign had occurred.

The measured reciprocal correlation times are plotted against Q in figure 2. It can immediately be seen, firstly, that the data are much less scattered than those reported previously (Mannella et al 1986) and secondly that, while good agreement with the JR predictions (full curve) is obtained for small Q, the data pass through a shallow minimum and start to rise again at larger values of Q. The latter effect is, however, much less pronounced than in the case of the earlier simulator.

In addition to the weak additive noise mentioned above, the electronic circuit (in common with all other real macroscopic physical systems) must also have an additive constant. Thus, in reality, (1) must be replaced by

$$\dot{x} = dx - bx^3 + \xi x + \zeta + g \tag{7}$$

where ζ represents noise of zero mean which, for convenience, we take to be Gaussian and white, with autocorrelation function

$$\langle \zeta(t)\zeta(t') \rangle = 2D\delta(t - t') \tag{8}$$

and cross-correlation function

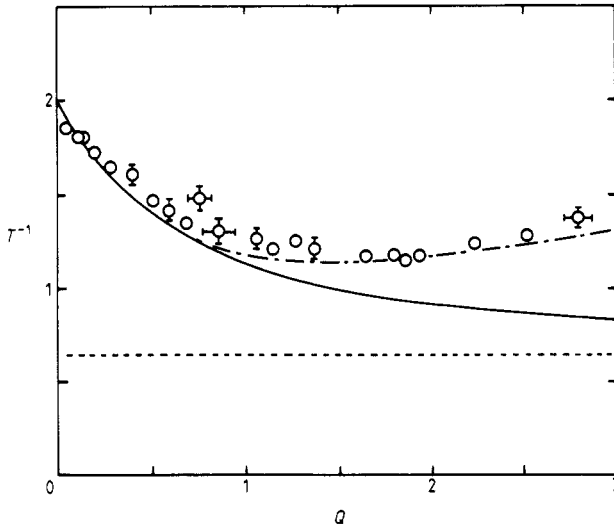$$\zeta(t)\xi(t') \rangle = 0 \tag{9}$$

Figure 2. Reciprocal relaxation times T^{-1} measured for the circuit shown in figure 1, as a function of the noise intensity Q (points) compared with the JR calculation (full curve). The broken line indicates the limiting $(Q \to \infty)$ value of T^{-1} calculated for equation (1). The chain curve shows $T^{-1}(Q)$ for equation (7) with $\zeta = 0$, $g = 5 \times 10^{-3}$.
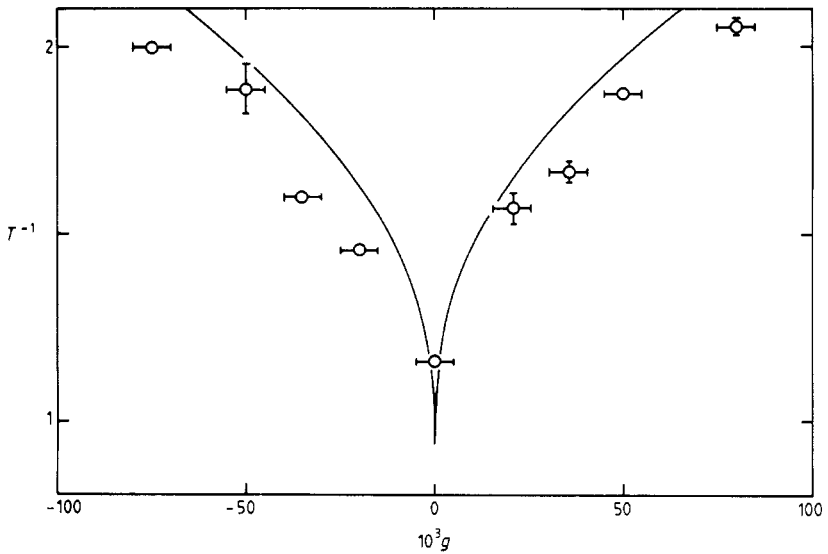

Figure 3. Reciprocal relaxation times T^{-1} measured for the circuit shown in figure 1, but with a variable small additive constant g added so as to model equation (7), for $Q = 1.89$ (points), compared with a theoretical value calculated by means of the JR algorithm (full curve).

and g is a constant. In practice, the circuit can be adjusted such that the magnitude of g is very small compared with the equilibrium values of x, $\pm(d/b)^{1/2}$, but it can never be set to a value identically equal to zero (Kondepudi *et al* 1986). The circuit will, of course, inevitably possess other non-idealities in addition to ζ and g; for example, small terms in x^n where $n > 3$. It could be argued that these terms, too, ought to be introduced on the right-hand side of (7). Such terms are, however, of much less significance than ζ and g precisely because they are not symmetry breaking, and thus should not have any dramatic effect. Furthermore, they will have their maximum influence when x is relatively large whereas, for the case of current interest with large Q, the probability density is strongly concentrated at small values of x.

We have made a systematic study of the effect on T of a small finite g. A set of T^{-1} data measured for the circuit with $Q = 1.89$ is shown in figure 3, where the full curve represents the result of a JR-like calculation in which the existence of a finite value of g was taken explicitly into account. In each case, $x > 0$ for $g > 0$ and $x < 0$ for $g < 0$. The quality of the agreement between experiment and theory is strikingly good. It is immediately clear, in the light of these results, why the T^{-1} data for the earlier circuit were both more scattered than the present data, and systematically high: the values of g in the former case will on average have been larger, because of the greater tendency of the prototype circuit to drift with time, exacerbated by the much longer data acquisition periods then required for the older data processor.

In § 5 we discuss in more detail the effect of the small but vitally important differences (non-zero ζ and g) between (1) and the real physical systems that it might have been expected to model.

3. Calculation of T by continued fraction expansion

We have already mentioned that the JR algorithm allows $T^{-1}(Q)$ for (1) to be calculated reliably for all Q. The method (Jung and Risken 1985) is applicable to any one-dimensional stochastic system driven by white noise, once the Liouvillian and the equilibrium distribution are known.

Prior to the introduction of the JR method, continued fraction expansions (CFE) were used (Hernández-Machado *et al* 1984, Faetti *et al* 1984) and, as already stated, it can now be seen that they yielded satisfactory results only for small values of Q. It is of some importance to establish the origins of the disagreement found at larger Q. This is because, firstly, there are a number of cases where the CFE method offers particular advantages as compared to the JR methods. In a one-dimensional system the analytical procedure introduced by JR provides by far the most superior algorithm for the calculation of T, but for multi-dimensional systems the procedure cannot be used. In the latter cases, the MCF technique of JR can of course still be applied, but it tends to be very greedy of computer CPU time as compared with CFE methods.

The second reason that the CFE analysis of the Stratonovich model is still worthy of investigation and further development relates to the wide application of CFE methods throughout solid state physics (Grosso and Pastori-Paravicini 1985) in, for example, calculations of densities of states. Only seldom is it possible in practice to carry out a full resummation of the continued fraction. For the particular case of (1), on the other hand, the complete solution has already been provided; the infinite-order resummation effectively provided by the JR results can thus be used as a guide to the effectiveness or otherwise of resummation techniques that can be applied to continued

fractions in many fields of physics. It should be noted in this context that Hong and Lee (1985) have also developed a very efficient CFE without truncation. The first few expansion parameters are determined via a rigorous calculation on the actual system under study; the remaining infinite expansion parameters are derived from an exactly solvable model, approximating the dynamical behaviour of that under study. The CFE of Lee and co-workers stems from the same generalised Langevin approach as that behind the methods reviewed by Grosso and Pastori-Parravicini (1985); see, for example Giordano *et al* (1981) and Lee (1982a, b, 1983).

Various explanations (Hernández-Machado *et al* 1984, Faetti *et al* 1984) for the earlier failure of the CFE method applied to (1) for large Q have been proposed, but none of them has led to improvements in the technique sufficient to yield correct values of T. Indeed, as we will show in a moment, the disagreement was due to an improper use of the CFE: using a correct version of the CFE, good qualitative agreement with JR may be obtained.

Let us first recall some basic ideas about the CFE (see also Faetti *et al* 1982). The relaxation time T of a correlation function is defined

$$T = \int_0^\infty \frac{\langle v(t)w \rangle}{\langle v(0)w \rangle} \, dt \tag{10}$$

which can be written as

$$T = \int_0^\infty \frac{\int v(t)w\rho_{eq}(\Gamma)d\Gamma}{\int v(0)w\rho_{eq}(\Gamma)d\Gamma} \, dt \tag{11}$$

or

$$T = \int_0^\infty \frac{\int \rho_{eq}(\Gamma)w \exp(L^\dagger t)v d\Gamma}{\int v(0)w\rho_{eq}(\Gamma)d\Gamma} \, dt \tag{12}$$

or

$$T = \int_0^\infty \frac{\int \rho_{eq}(\Gamma)w\rho \exp[t(L^\dagger + i\omega)]v d\Gamma}{\int v(0)w\rho_{eq}(\Gamma)d\Gamma} \, dt \Bigg|_{\omega=0} \tag{13}$$

whence

$$T = \frac{1}{\int v(0)w\rho_{eq}(\Gamma)d\Gamma} \int \rho_{eq}(\Gamma)w \frac{1}{L^\dagger + i\omega} v \, d\Gamma \Bigg|_{\omega=0}. \tag{14}$$

In the equations (10)-(14), ρ_{eq} is the equilibrium distribution, and v and w are two generic state variables. L is the Liouvillian for the Stratonovich model,

$$L = -\partial_x(x - x^3) + Q\partial_x x\partial_x x. \tag{15}$$

This is where the CFE method is now applied. The integral

$$\int \rho_{eq}(\Gamma)w \frac{1}{L^\dagger + i\omega} v \, d\Gamma$$

is expanded as

$$\cfrac{1}{-i\omega + \gamma_0 - \cfrac{K_1}{-i\omega + \gamma_1 - \cfrac{K_2}{\ddots}}} \Bigg|_{\omega=0} \tag{16}$$

Explicit expressions for the first two γ_i and K_i may be found in Hernández-Machado *et al* (1984) (but note the differing sign of K_i). The general forms for γ_i and K_i may

be obtained via the moments of $\langle (L^{\dagger})^n \rangle$ by use of a suitable algorithm (see Grigolini *et al* 1983, Grosso and Pastori-Parravicini 1985) where $\langle (L^{\dagger})^n \rangle \equiv \langle w|(L^{\dagger})^n|v \rangle$. In other words, there is a mapping from $\{\langle (L^{\dagger})^0 \rangle, \ldots, \langle (L^{\dagger})^{2n+1} \rangle \}$ to $\{\gamma_0, \ldots, \gamma_n, K_1, \ldots, K_{n+1}\}$. This implies that all the information incorporated in (16) is already contained in a mere expansion at short times of the correlation function (12). Thus, assuming that

$$\int w\rho_{eq}(\Gamma)v(0)d\Gamma = 1$$

we can make the formal statement that

$$(16)|_{\omega=0} = \int_0^{\infty} \rho_{eq}(\Gamma)w \exp(L^{\dagger}t)v \, d\Gamma \, dt \qquad (\text{order } n)$$

$$= F(\omega, (L^{\dagger})^0, \ldots, (L^{\dagger})^{n-1})|_{\omega=0}.$$

However,

$$\langle (L^{\dagger})^n \rangle = \partial_{t^n}^n \int \rho_{eq}(\Gamma)w \exp(L^{\dagger}t)v \, d\Gamma \bigg|_{t=0}$$

which is the coefficient of a Taylor-like expansion for short times. It is plain, therefore, that what a CFE will always do is to give a value to (16) based only on the short timescale. For the Stratonovich model, on the other hand, the fluctuations of the system become slower and slower as Q increases, posing a problem for the convergency of (16). It is clear, therefore, that any sensible result must be based upon an infinite resummation of (16). The MCF of JR, on the other hand, solves this problem, working directly towards the building of $L^{\dagger-1}$ (note that $T = \int w\rho_{eq}(\Gamma)(L^{\dagger})^{-1}v \, d\Gamma$). Developing such a resummation of (16), we compute the parameters γ_i, K_i by use of the Mori-Lanczós algorithm (Saad 1982, Grosso and Pastori-Parravicini 1985).

The next step is to study the asymptotic behaviour of these two quantities: similar procedures can be found in the literatures applied to the Lorentz model (Grossman and Sonneborn-Schmick 1982) and the Duffing oscillator (Fronzoni *et al* 1985). The idea is to find a function whose CFE is similar, at least asymptotically, to the CFE that we are dealing with. In the event, we found (see figures 4(a) and (b)) that in the
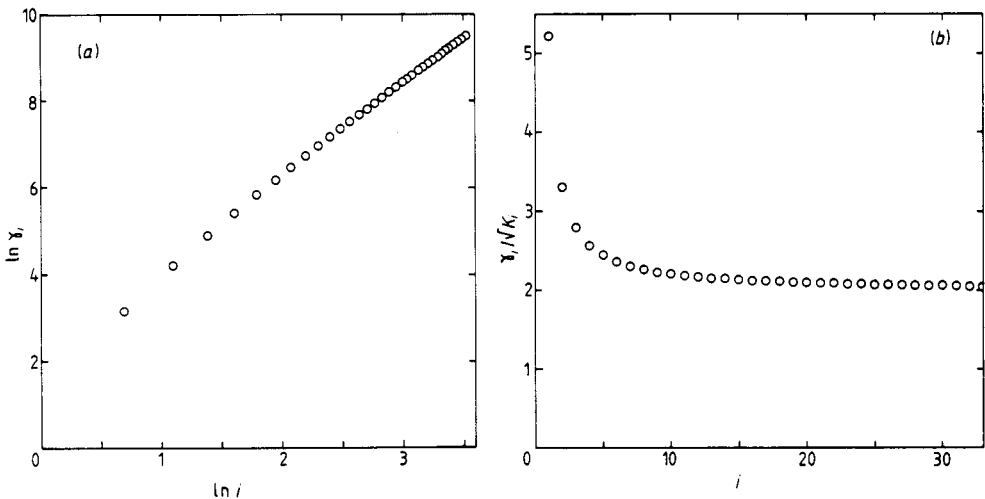

Figure 4. Asymptotic behaviour of some quantities in the CFE calculation of the relaxation time for $Q = 1.5$: (a) variation of $\ln \gamma_i$ with $\ln i$; (b) variation of $\gamma_i/\sqrt{K_i}$ with i.

asymptotic limit of $i \to \infty$

$$\lim_{i \to \infty} \gamma_i \propto (i-1)^2 \qquad \lim_{i \to \infty} K_i \propto i^4$$

and that

$$\lim_{i \to \infty} \gamma_i / \sqrt{K_i} \sim 2.$$

This is identical to a known CFE expansion (Wall 1967, where on p 370 we take the limit a, $b \to 0$ in equation (94.3)), namely

$$\int_0^\infty \frac{cx^2}{\sinh(cx)} \, \mathrm{d}x = \cfrac{1}{b_1 - \cfrac{a_1}{b_2 - \cfrac{a_2}{\ddots}}} \tag{17}$$

where

$$b_{i+1} = (2i^2 + 2i + 1)c^2 \tag{17a}$$

and

$$a_i = i^4 c^4 \frac{4i^2}{4i^2 - 1}. \tag{17b}$$

The procedure that is now adopted (Fronzoni *et al* 1985) is, firstly, to compute γ_i, K_i up to n and, secondly, to compute a_i, b_i up to n. Then

$$\int_0^\infty \frac{cx^2}{\sinh(cx)} \, \mathrm{d}x = \cfrac{1}{b_1 - \cfrac{a_1}{b_2 - \cfrac{a_2}{\ddots \cfrac{a_{n-1}}{b_n - \mathrm{tail}(n)}}}} \tag{18}$$

and

$$(16)|_{\omega=0} = T = \cfrac{1}{\gamma_0 - \cfrac{K_1}{\gamma_1 - \cfrac{K_2}{\ddots \cfrac{K_{n-1}}{\gamma_{n-1} - \mathrm{tail}'(n)}}}}. \tag{19}$$

Thirdly, we argue that $\mathrm{tail}(n) = \mathrm{tail}'(n)$ because of the similar asymptotic behaviour.

The constant c in (18) is determined on the assumption that $a_{n-1} = K_{n-1}$, or $b_n = \gamma_{n-1}$, since in the limit $i \to \infty$ we require that both must give the same value of c. Thus, computing the integral in (18) by simple quadrature, and solving (18) for $\mathrm{tail}(n)$, we can compute $\mathrm{tail}'(n)$ for (19) and hence determine the value of T.

We have used this procedure to evaluate $T(n)$, the computed value of T in which the proper tail, evaluated as described above, has been added to the nth step. In doing so we have found that, even for $n = 32$, representing the largest number of CFE steps that we are able to compute, $T(n)$ is still weakly dependent on n as shown in figure 5. We cannot, therefore, expect that $T(32)$ will correspond to a quantitatively accurate value of the relaxation time but we can hope for a reasonably close approximation. Values of $T^{-1}(32)$, computed in this way, are plotted as a function of Q in figure 6. It is immediately apparent that excellent qualitative agreement with JR has been
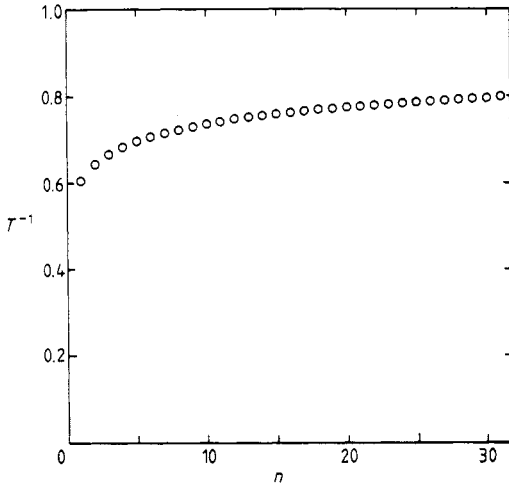

Figure 5. The reciprocal relaxation time T^{-1} at $Q = 1.5$ calculated by use of the asymptotic resummation (tail) at step n, as a function of n. The exact (JR) value of T^{-1} at this Q is 1.0.
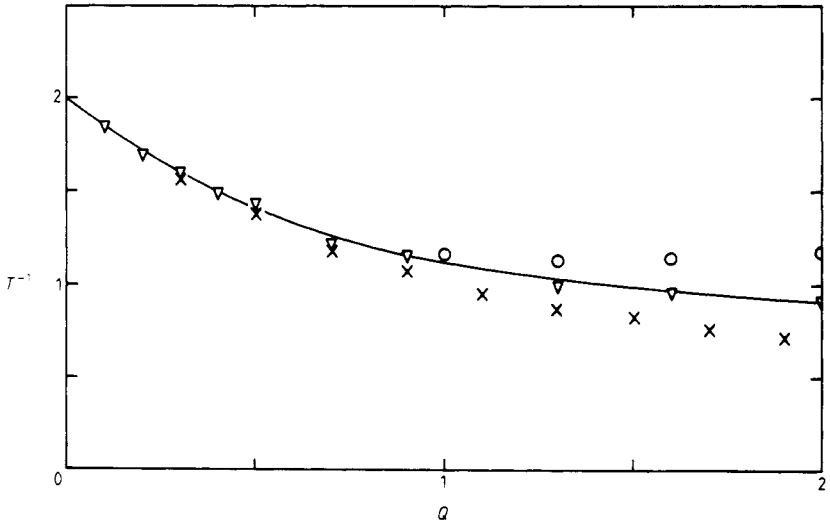

Figure 6. Reciprocal relaxation times T^{-1} as functions of noise intensity Q determined by CFE calculation (crosses) and by digital simulation (triangles) for comparison with the exact (JR) result shown by the full curve. Also shown (circles) are the $T^{-1}(Q)$ results computed by digital simulation for the case where there is a small additive constant of 5×10^{-3}.

obtained, even though the absolute magnitude of T^{-1} for large Q is too low by about 15%. This result represents a substantial improvement over the earlier CFE calculations of T^{-1}, which always tended to pass through a minimum and rise again at large Q, quite contrary to the JR calculation shown by the full curve in figure 6.

4. Computation of T by digital simulation

In this section we briefly describe the computer simulation of the Stratonovich model. We tried two different algorithms: one is the algorithm described by Sancho *et al* (1982a), the other is a high-order Adam predictor-corrector (Lapidus and Seinfeld 1971). Both gave consistent results, even if the latter suffered from slowness in the rate of convergence. It had been argued (Sancho 1986, Straub 1986) that the stochastic nature of the equation we are trying numerically to integrate may lead to mistakes when using higher-order predictor-corrector methods which, on the contrary, rely on the assumption of smooth trajectories. The data presented here were computed via the algorithm of Sancho *et al* (1982a). There are some differences, however, as regards the method we used to compute T^{-1}. Firstly, we computed the correlation function via a standard FFT technique. Secondly, the average was taken over typically 100 blocks (a factor of four bigger than the number of blocks used by Sancho *et al*). Thirdly, we applied the small baseline corrections, described by Sancho *et al* (1985), to the averaged correlation function. It is important to stress that this procedure is exactly the same as is followed in the analogue simulation case. The results are plotted (for the additive constant equal to zero) in figure 6 as crosses, while, still in figure 5, the circles represent the results for a small additive constant ($=5 \times 10^{-3}$). The agreement between digital data and JR theory is strikingly good. The integration time step was 1×10^{-3}, and the FFT was typically computed using an array of 10^4 values of x.

5. Discussion

The results presented above demonstrate that calculations of $T^{-1}(Q)$ for (1) by CFE and by digital simulation are in satisfactory agreement with JR, the 15% departure of the CFE from JR at large Q being attributable to an insufficient number of terms having been taken in the expansion. It is both satisfying and reassuring that calculations of T^{-1} by three such very different methods should have yielded consistent results. The values of $T^{-1}(Q)$ measured for the analogue electronic circuit (figure 2) are, however, qualitatively different from those predicted, in that the value of T^{-1} passes through a broad minimum and then starts to rise again at large Q. We have already suggested in § 2 that the discrepancy arises on account of the considerations first put forward by Brand (1984), namely that a real system can never by described by the original idealised Stratonovich model (1), but is likely to be describable instead in terms of an equation of the form of (7). In the present section we take a little further the discussion of the discrepancy in terms of differences between 'real physical systems', of which the electronic circuit is an example, and the idealised equation such as (1) which have been used to model them.

We should start by noting that our assumption (following Fedchenia and Usova (1983) and Brand (1984)) that the additive noise term ζ in (7) is Gaussian and white must certainly be in error. It is much more likely, in fact, that the noise will be peaked

at some particular frequency. Nonetheless, for the purposes of the present discussion we will retain these simplifying assumptions. One of the most important consequences of a non-zero D in (8) is that transitions will occur occasionally between the two different accessible regions $x \lessgtr 0$ and, as already noted above, such transitions did indeed occur infrequently in the actual circuit, at intervals of about 10^2 s. This number is to be compared to the deterministic relaxation time for the system of $T = 0.5$ s at $Q = D = 0$. For the extremely small typical additive noise intensities found in the circuit on the assumption that $\langle x^2 \rangle \simeq D$ ($D \simeq 10^{-5}$ for $Q = 0$, but it is likely to be dependent on Q), we find that typical values of T^{-1} are 2.3×10^{-2} s at $Q = 2.44$ and 7.3×10^{-2} s at $Q = 3.37$. To measure the correlation times for such slow relaxations it was, of course, necessary to observe the system continuously for very long times, typically of 5×10^2 s.

It is clear that in a real system of the Stratonovich type there are two very different timescales: the first is related to motion inside one of the potential wells and is fast, and the second relates to hopping between the wells and, for small values of D, is exceptionally slow. Because of the huge timescale difference, it is reasonable to study the correlation time for the case where the system is confined to one well: this is the justification (see § 2) for our having programmed the data processor to discard any measured $x(t)$ sequences which changed sign during the period of observation. For data acquired and processed in this way, such that the effect of inter-well hopping is effectively eliminated, we may expect the system to be described by (7)–(9) with $D = 0$.

The Fokker–Planck equation corresponding to (7) is

$$\partial_t P(x, t) = [-\partial_x(x - x^3 + g) + Q\partial_x x \partial_x x]P(x, t) \tag{20}$$

which yields the equilibrium distribution

$$P_{\text{st}}(x, g) = N|x|^{1/Q-1} \exp\left[-\frac{1}{Q}\left(\frac{x^2}{2} + \frac{g}{x}\right) \right] \tag{21}$$

where N is the appropriate normalisation constant. In relation to (21), it should be understood that x must be restricted to positive values when $g > 0$. This is because, for $g > 0$, $P_{\text{st}}(x, g)$ diverges for $x \to 0_-$. No finite normalisation is possible, therefore, and we must accordingly disregard the negative part of the x axis. A similar argument applies to the case of $g < 0$, when we must restrict x to negative values. In physical terms, these restrictions relate to the fact that the equilibrium distribution must correspond to the system occupying the deeper of the two potential wells and to the vanishing of the probability current at $x = 0$.

One of the most interesting consequences of the finite value of g in (21) is that

$$\lim_{x \to 0_+} P_{\text{st}}(x, g > 0) \equiv 0 \tag{22}$$

for finite Q. This is in marked contrast to

$$\lim_{x \to 0} P_{\text{st}}(x, 0) \sim |x|^{1/Q-1} \tag{23}$$

for the idealised equation (1) with $g = 0$. Furthermore, $P_{\text{st}}(x, g)$ is peaked for large Q and small g at

$$x_m(g \neq 0) \sim g/Q \tag{24}$$

whereas $P_{\text{st}}(x, 0)$ for $Q > 1$ has a maximum at

$$x_m(g = 0) = 0. \tag{25}$$

We are now in a position to discuss qualitatively the physical difference between the two cases $g = 0$ and $g \neq 0$. The relaxation time is determined by the diffusion coefficient. In other words, as the diffusion coefficient becomes smaller we may expect the correlation time to increase. We consider the evolution of $\langle x \rangle$. This quantity is driven by $L^{\dagger}$, with

$$L^{\dagger} = (x - x^3 + Qx + g)\partial_x + Qx^2\partial_{x^2}^2. \tag{26}$$

If $g = 0$, we already know that T^{-1} decreases with increasing Q. This behaviour must somehow be reflected in the diffusive term in (26): the form of $T^{-1}(Q)$ must be determined by the joint action of an increase in Q and a decrease in the most probable value of x. For large Q, $x_m = 0$ and the probability distribution itself is squeezed more and more towards zero. Physically, the system becomes 'stuck' near the origin ($x = 0$) where it is then unresponsive to the noise because the latter enters (1) multiplicatively. In practice, the combined effect is a decrease in the diffusion coefficient and an increase in T.

The behaviour to be expected when $g \neq 0$ is very different. Under these conditions, the system is prevented from converging towards $x = 0$ and, for any Q, $P_{\text{st}}(0_+, g > 0) = 0$. Because the most probable value of x is now different from zero, we would expect that an increase of Q will force the system into more rapid motion, with a corresponding decrease of T as found in the analogue experiment.

In order to test these ideas, we have computed T^{-1} by use of the JR algorithm (their equation (2.43)), but applied to (7) rather than (1). Firstly we set $Q = 1.89$, $D = 0$ and studied the effect of a variation in g, yielding the full curve of figure 3. A comparison with the analogue data implies that in practice the average magnitude of the additive constant g is of the order of a millivolt. We have also calculated T^{-1} as a function of Q for $g = 5 \times 10^{-3}$ V, with the result shown by the chain curve in figure 2. Given that the precise value of g for the circuit will have varied with time, and perhaps also to a small extent with Q, the agreement with the analogue data can be regarded as remarkably good.

6. Conclusion

We have carried out a detailed study of the relaxation time for the Stratonovich model (1), using three entirely different approaches. An enhanced CFE method has been presented which yields a $T^{-1}(Q)$ characteristic that is close to the JR curve: it would, we believe, have been in complete agreement had it been possible to evaluate more terms in the expansion. A new digital simulation of (1) has been carried out and shown to be in excellent agreement with JR.

Measurements of $T^{-1}(Q)$ for an analogue electronic circuit built to model (1) are in striking disagreement, however, both quantitatively and qualitatively, with the JR prediction; but they are not inconsistent with earlier analogue measurements at very large Q. The discrepancies are attributable to the fact that real physical systems, as exemplified by the electronic circuits, will in reality be described not by (1), but by (7), because there will invariably be some additive noise and an additive constant. A finite additive noise, no matter how small, will induce transitions between the two wells, an effect that is entirely absent from (1); in the experiments we eliminated the influence on T^{-1} of this phenomenon by a deliberate restriction to single-sign $x(t)$ blocks during data acquisition. A finite additive constant, no matter how small, will

always cause $T^{-1}(Q)$ to pass through a minimum and rise again at large enough Q. Such behaviour is to be anticipated even for those physical systems that, at first sight, appear to be quite accurately approximated by (1).

Acknowledgments

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